Fact: Let $g(x) = \begin{cases} 1-|x|, & |x| \le 1, \\ 0, & \text{otherwise}. \end{cases}$

Then $\hat{g}(\xi) = \left(\dfrac{\sin \pi\xi}{\pi\xi}\right)^2$.

Ch5 Ex 2

1. Let $\tilde{\mathcal{F}}_R(t)$ be the Fejér kernel on the real line, i.e. $\tilde{\mathcal{F}}_R(t) = R\left(\dfrac{\sin \pi t R}{\pi t R}\right)^2$.

Let $F_N(x)$ be the Fejér kernel of 1-periodic function, i.e.

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}.$$

Then $F_N(x) = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{F}}_N(x+n)$.

Pf: Recall Poisson Summation Formula,

if $f, \hat{f} \in \mathcal{M}(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Fourier Inversion Formula.

Then
$$\sum_{n=-\infty}^{\infty} \hat{f}(x+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n x}.$$

Let
$$h_R(x) = g\left(\frac{x}{R}\right) = \begin{cases} 1 - \frac{|x|}{R}, & |x| < R \\ 0, & \text{otherwise.} \end{cases}$$

Then
$$\hat{h}_R(\xi) = R\,\hat{g}(R\xi) = R\left(\frac{\sin \pi R\xi}{\pi R\xi}\right)^2.$$

By Poisson Summation Formula,

$$\sum_{n=-\infty}^{\infty} \hat{h}_N(x+n) = \sum_{n=-\infty}^{\infty} h_N(n) e^{-2\pi i n x}.$$

$$\sum_{n=-\infty}^{\infty} \hat{h}_N(x+n) = \sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n)$$

$$\sum_{n=-\infty}^{\infty} h_N(n) e^{-2\pi i n x} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{-2\pi i n x} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = F_N(x)$$

$\square$

2. $f \in \mathcal{M}(\mathbb{R})$, $\operatorname{supp} \hat{f} = I$ where $I = [-\frac{1}{2}, \frac{1}{2}]$

(a) Prove $f(x) = \sum_{n=-\infty}^{\infty} f(n) K(x-n)$ where $K(y) = \dfrac{\sin \pi y}{\pi y}$.

Pf: By Poisson Summation Formula,

$$\sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}, \quad \forall \xi \in \mathbb{R}.$$

Since $\operatorname{supp} \hat{f} = I$, when $\xi \in I$,

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}.$$

When $\xi \notin I$, $\hat{f}(\xi) = 0$.

For any $\xi \in \mathbb{R}$,

$$\hat{f}(\xi) = \chi_I \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}.$$

By Fourier Inversion Formula,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$= \sum_{n=-\infty}^{\infty} \int \chi_I f(n) e^{-2\pi i n \xi} e^{2\pi i x \xi} \, d\xi$$

$$= \sum_{n=-\infty}^{\infty} f(n) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (x-n)\xi} \, d\xi$$

$$= \sum_{n=-\infty}^{\infty} f(n) \frac{e^{\pi i (x-n)} - e^{-\pi i (x-n)}}{2\pi i (x-n)}$$

$$= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}$$

$$= \sum_{n=-\infty}^{\infty} f(n) K(x-n) .$$

$\square$

(b) Let $\lambda > 1$. Prove

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right) \text{ where } K_\lambda(y) = \frac{\cos \pi y - \cos \pi \lambda y}{\pi^2 \lambda (\lambda - 1) y^2}.$$

Pf: Let $f_\lambda(x) = f\left(\frac{x}{\lambda}\right)$.

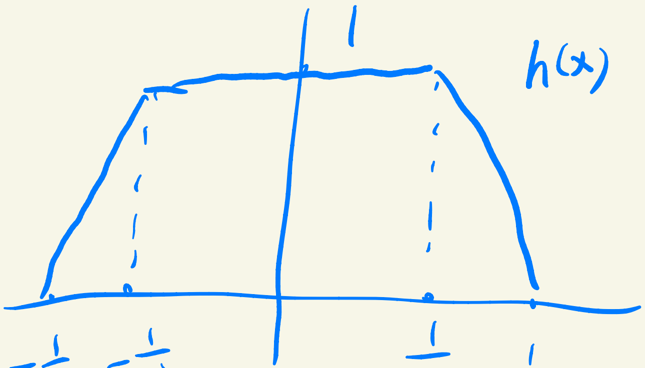
Then $\hat{f_\lambda}(\xi) = \lambda \hat{f}(\lambda \xi)$.

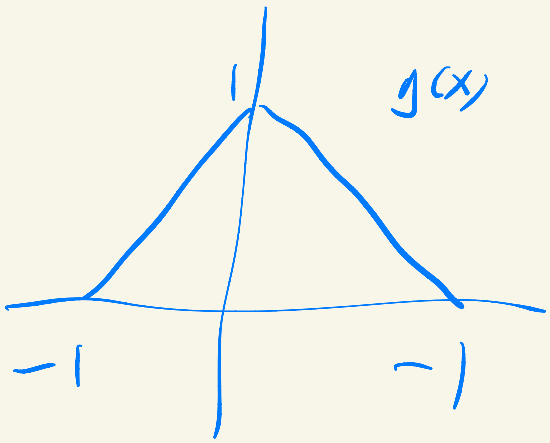
Since $\operatorname{supp} \hat{f} = I$, $\operatorname{supp} \hat{f_\lambda} = \left[-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right] \subset I$

As before, by Poisson Summation formula,

$$\hat{f}_\lambda(\xi) = \chi_I \sum_{n=-\infty}^{\infty} f_\lambda(n) e^{-2\pi i n \xi}$$

$$\chi_{[-\frac{1}{2\lambda}, \frac{1}{2\lambda}]}$$





$$\hat{f}_\lambda(\xi) = \hat{f}_\lambda(\xi) h(\xi)$$

$$= h(\xi) \chi_I(\xi) \sum_{n=-\infty}^{\infty} \hat{f}_\lambda(n) e^{-2\pi i n \xi}$$

$$= h(\xi) \sum_{n=-\infty}^{\infty} \hat{f}_\lambda(n) e^{-2\pi i n \xi}$$

By Fourier Inversion Formula,

$$f\left(\frac{x}{\lambda}\right) = f_\lambda(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) \int_{-\infty}^{\infty} h(\xi) e^{-2\pi i \xi (n-x)} d\xi$$

$$= \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) \hat{h}(n-x)$$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) \hat{h}(n - \lambda x) .$$

Note that

$$h(x) = \frac{\lambda}{\lambda-1} g(2x) - \frac{1}{\lambda-1} g(2\lambda x) .$$

$$\hat{h}(x) = \frac{\lambda}{\lambda-1} \frac{1}{2} \hat{g}\left(\frac{x}{2}\right) - \frac{1}{\lambda-1} \frac{1}{2\lambda} \hat{g}\left(\frac{x}{2\lambda}\right)$$

$$= \frac{1}{2\lambda(\lambda-1)} \left[\lambda^2 \left(\frac{\sin \frac{\pi x}{2}}{\frac{\pi x}{2}} \right)^2 - \left(\frac{\sin \frac{\pi x}{2\lambda}}{\frac{\pi x}{2\lambda}} \right)^2 \right]$$

$$= \frac{1}{2\lambda(\lambda-1)} \frac{4\lambda^2}{\pi^2 x^2} \left[\left(\sin \frac{\pi x}{2} \right)^2 - \left(\sin \frac{\pi x}{2\lambda} \right)^2 \right]$$

$$= \frac{2\lambda}{\pi^2 (\lambda-1) x^2} \left(\frac{1-\cos \pi x}{2} - \frac{1-\cos \frac{\pi x}{\lambda}}{2} \right)$$

$$= \frac{\lambda}{\pi^2 (\lambda-1) x^2} \left(\cos \frac{\pi x}{\lambda} - \cos \pi x \right)$$

$$\hat{h}(n-\lambda x) = \frac{\cancel{\lambda}}{\pi^2(\lambda-1)\lambda^{\cancel{4}}\left(\frac{n}{\lambda}-x\right)^2}\left(\cos\pi\left(\frac{n}{\lambda}-x\right)-\cos\pi\lambda\left(\frac{n}{\lambda}-x\right)\right)$$

$$= K_\lambda\left(\frac{n}{\lambda}-x\right).$$

Hence.

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) K_\lambda\left(\frac{n}{\lambda}-x\right).$$

$\square$

(c) $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f(n)|^2.$

Recall in (a),

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}, \quad \forall \xi \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

$\hat{\varphi}(n)$ $\quad \Rightarrow \varphi$ 1-periodic

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

Parseval's Identity

$$= \int_{-\infty}^{\infty} |f(\xi)|^2 d\xi$$

Plancherel Formula.

$\square$